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Generalization of Hermite functions by fractal interpolation

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Abstract

Fractal interpolation techniques provide good deterministic representations of complex phenomena. This paper approaches the Hermite interpolation using fractal procedures. This problem prescribes at each support abscissa not only the value of a function but also its first p derivatives. It is shown here that the proposed fractal interpolation function and its first p derivatives are good approximations of the corresponding derivatives of the original function. According to the theorems, the described method allows to interpolate, with arbitrary accuracy, a smooth function with derivatives prescribed on a set of points. The functions solving this problem generalize the Hermite osculatory polynomials.
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1. Introduction

Fractal interpolation techniques provide good deterministic representations of complex phenomena such as economic time series, weather data, etc. The main difference with classic procedures consists in the definition of a functional relation assuming a self-similarity on small scales [1,2]. The theorem of Barnsley and Harrington [3, Theorem 2] proves the

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existence of differentiable fractal interpolation functions. This kind of approximants can generalize the piecewise polynomial interpolation as for instance, the method of spline functions [7]. This paper approaches the Hermite interpolation using fractal procedures. This problem prescribes at each support abscissa not only the value of a function but also its first p derivatives. The original and the reconstructed functions thus have a contact of order p at the nodes. The fractal interpolation functions solving this problem contain the Hermite osculatory polynomials as a particular case.

In the second part of the paper, the uniform distance between a smooth original function and the proposed fractal interpolation function is studied. The results obtained prove that the first derivatives are good approximations of the corresponding derivatives of the function. As a consequence, if a sequence of interval partitions Δ_m such that $\|\Delta_m\| \to 0$ is considered, the error of interpolation approaches zero.

2. Generalization of the Hermite functions by fractal interpolation

2.1. Hermite functions

Given a partition $\Delta: t_0 < t_1 < \cdots < t_N$ of an interval $[t_0, t_N]$, $I_n = [t_{n-1}, t_n]$ for $1 \leqslant n \leqslant N$, the Hermite function space [8] H_Δ^{p+1} $(p \in \mathbb{N})$ is defined by

$$H_\Delta^{p+1} = \{\varphi : [t_0, t_N] \to \mathbb{R};\ \varphi \in \mathcal{C}^p[t_0, t_N], \quad \varphi|_{I_n} \in \mathcal{P}_{2p+1}\},$$

where $\mathcal{P}_{2p+1}$ is the space consisting of all polynomials of degree at most $2p+1$.

In order to approximate a given real function $x \in \mathcal{C}^p[t_0, t_N]$ by a function $\varphi \in H_\Delta^{p+1}$ the component polynomials $p_n = \varphi|_{I_n}$ are chosen so that $p_n \in \mathcal{P}_{2p+1}$ and for $0 \leqslant k \leqslant p$:

$$p_n^{(k)}(t_{n-1}) = x^{(k)}(t_{n-1}), \quad p_n^{(k)}(t_n) = x^{(k)}(t_n).$$

The existence of a unique solution for this problem is guaranteed [8].

2.2. Fractal interpolation functions

Let $t_0 < t_1 < \cdots < t_N$ be real numbers, and $I = [t_0, t_N] \subset \mathbb{R}$ the closed interval that contains them. Let a set of data points $\{(t_n, x_n) \in I \times \mathbb{R} : n = 0, 1, 2, \ldots, N\}$ be given. Set $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n,\ n \in \{1, 2, \ldots, N\}$ be contractive homeomorphisms such that

$$L_n(t_0) = t_{n-1},\ L_n(t_N) = t_n, \tag{1}$$

$$|L_n(c_1) - L_n(c_2)| \leqslant l\ |c_1 - c_2| \quad \forall c_1, c_2 \in I \tag{2}$$

for some $0 \leqslant l < 1$.

Let $-1 < \alpha_n < 1$; $n = 1, 2, \ldots, N$, $F = I \times [c, d]$ for some $-\infty < c < d < +\infty$ and N continuous mappings, $F_n : F \to \mathbb{R}$ be given satisfying

$$F_n(t_0, x_0) = x_{n-1}, \quad F_n(t_N, x_N) = x_n, \quad n = 1, 2, \ldots, N, \tag{3}$$

$$|F_n(t, x) - F_n(t, y)| \leqslant \alpha_n |x - y|, \quad t \in I, \quad x, y \in \mathbb{R}. \tag{4}$$

Now define functions $w_n(t,x) = (L_n(t), F_n(t,x)),\ \forall n = 1, 2, ..., N$.

Theorem 1 (*Barnsley [1,2]*). *The iterated function system* (IFS) [5] $\{F,\ w_n : n = 1, 2, ..., N\}$ *defined above admits a unique attractor G. G is the graph of a continuous function* $f : I \to \mathbb{R}$ *which obeys* $f(t_n) = x_n$ *for* $n = 0, 1, 2, ..., N$.

The previous function is called a fractal interpolation function (FIF) corresponding to $\{(L_n(t), F_n(t,x))\}_{n=1}^N$. $f : I \to \mathbb{R}$, is the unique function satisfying the functional equation

$$f(L_n(t)) = F_n(t, f(t)), \quad n = 1, 2, ..., N, \quad t \in I$$

or

$$f(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)), \quad n = 1, 2, ..., N, \quad t \in I_n = [t_{n-1}, t_n]. \tag{5}$$

Let $\mathcal{F}$ be the set of continuous functions $f : [t_0, t_N] \to [c, d]$ such that $f(t_0) = x_0$; $f(t_N) = x_N$. Define a metric on $\mathcal{F}$ by

$$\|f - g\|_\infty = \max\{|f(t) - g(t)| : t \in [t_0, t_N]\} \quad \forall f, \quad g \in \mathcal{F}$$

Then $(\mathcal{F}, d)$ is a complete metric space.

Define a mapping $T : \mathcal{F} \to \mathcal{F}$ by

$$(Tf)(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)) \quad \forall t \in [t_{n-1}, t_n], \quad n = 1, 2, ..., N.$$

Using (1)–(4), it can be proved that $(Tf)(t)$ is continuous on the interval $[t_{n-1}, t_n]$ for $n = 1, 2, ..., N$ and at each of the points $t_1, t_2, ..., t_{N-1}$. T is a contraction mapping on the metric space $(\mathcal{F}, d)$

$$\|Tf - Tg\|_\infty \leqslant |\boldsymbol{\alpha}|_\infty \|f - g\|_\infty, \tag{6}$$

where $|\boldsymbol{\alpha}|_\infty = \max\{|\alpha_n|;\ n = 1, 2, ..., N\}$. Since $|\boldsymbol{\alpha}|_\infty < 1$, T possesses a unique fixed point on $\mathcal{F}$, that is to say, there is $f \in \mathcal{F}$ such that $(Tf)(t) = f(t)\ \ \forall t \in [t_0, t_N]$. This function is the FIF corresponding to w_n.

The most widely studied fractal interpolation functions so far are defined by the IFS

$$\begin{cases} L_n(t) = a_n t + b_n, \\ F_n(t,x) = \alpha_n x + q_n(t), \end{cases} \tag{7}$$

where $q_n(t)$ is a polynomial [2,6]. α_n is called a vertical scaling factor of the transformation w_n.

2.3. Hermite fractal interpolation functions

The following theorem assures the existence of differentiable FIF.

Theorem 2 (*Barnsley and Harrington [3]*). *Let* $t_0 < t_1 < t_2 < \cdots < t_N$ *and* $L_n(t)$, $n = 1, 2, ..., N$, *the affine function* $L_n(t) = a_n t + b_n$ *satisfying the expressions* (1)–(2).

Let $a_n = L'_n(t) = \frac{t_n - t_{n-1}}{t_N - t_0}$ *and* $F_n(t,x) = \alpha_n x + q_n(t)$, $n = 1, 2, \ldots, N$ *verifying* (3)–(4). *Suppose for some integer* $p \geqslant 0$, $|\alpha_n| < a_n^p$ *and* $q_n \in C^p[t_0, t_N]$; $n = 1, 2, \ldots, N$. *Let*

$$F_{nk}(t,x) = \frac{\alpha_n x + q_n^{(k)}(t)}{a_n^k}, \quad k = 1, 2, \ldots, p, \tag{8}$$

$$x_{0,k} = \frac{q_1^{(k)}(t_0)}{a_1^k - \alpha_1} \quad x_{N,k} = \frac{q_N^{(k)}(t_N)}{a_N^k - \alpha_N}, \quad k = 1, 2, \ldots, p.$$

If

$$F_{n-1,k}(t_N, x_{N,k}) = F_{nk}(t_0, x_{0,k}) \tag{9}$$

with $n = 2, 3, \ldots, N$ *and* $k = 1, 2, \ldots, p$, *then* $\{(L_n(t), F_n(t,x))\}_{n=1}^N$ *determines a* FIF $f \in C^p[t_0, t_N]$ *and* $f^{(k)}$ *is the FIF determined by* $\{(L_n(t), F_{nk}(t,x))\}_{n=1}^N$, *for* $k = 1, 2, \ldots, p$.

The above result leads us to expect that the Hermite fractal interpolation problem can be solved uniquely. The following theorem guarantees the existence of a FIF with $p+1$ derivative values prescribed at the abscissas $((t_n, x_{nk});\ n = 0, 1, \ldots, N;\ k = 0, 1, \ldots, p)$.

Theorem 3. *Let* $N \geqslant 1$, $p \in \mathbb{N}$, $t_0 < t_1 < \cdots < t_N$ *and* $\{x_{nk}; n = 0, 1, \ldots, N;\ k = 0, 1, \ldots, p\}$ *be given. Let* $\alpha_1, \alpha_2, \ldots, \alpha_N$ *real numbers such that* $|\alpha_n| < a_n^p$ $\forall n = 1, 2, \ldots, N$, *with* $a_n = \frac{t_n - t_{n-1}}{t_N - t_0}$. *There exists precisely one function of fractal interpolation* $f \in \mathcal{C}^p$ *defined by an IFS given by*

$$\begin{cases} L_n(t) = a_n t + b_n, \\ F_n(t,x) = \alpha_n x + q_n(t), \end{cases} \tag{10}$$

where $q_n(t)$ $\forall n = 1, 2, \ldots, N$ *are polynomials of degree at most* $2p+1$, *such that* $f^{(k)}(t_n) = x_{nk}$ *for* $n = 0, 1, \ldots, N$; $k = 0, 1, \ldots, p$.

Proof. Consider

$$a_n = \frac{t_n - t_{n-1}}{t_N - t_0}, \quad b_n = \frac{t_N t_{n-1} - t_0 t_n}{t_N - t_0} \tag{11}$$

and define, for $0 \leqslant k \leqslant p$

$$F_{nk}(t,x) = \frac{\alpha_n x + q_n^{(k)}(t)}{a_n^k} \tag{12}$$

with $deg(q_n) = 2p+1$.

The polynomial $q_n(t)$ is computed as solution of the system of equations $(0 \leqslant k \leqslant p)$

$$\begin{cases} F_{nk}(t_0, x_{0k}) = \frac{\alpha_n x_{0k} + q_n^{(k)}(t_0)}{a_n^k} = x_{n-1,k}, \\ F_{nk}(t_N, x_{Nk}) = \frac{\alpha_n x_{Nk} + q_n^{(k)}(t_N)}{a_n^k} = x_{nk}. \end{cases} \tag{13}$$

The coefficients of $q_n(t)$ are the $2p+2$ unknowns of the above equations. The expressions (13) can also be written as

$$\begin{cases} (q_n \circ L_n^{-1})^{(k)}(t_{n-1}) = \frac{1}{a_n^k}\, q_n^{(k)}(t_0) = x_{n-1,k} - \frac{\alpha_n x_{0k}}{a_n^k}, \\ (q_n \circ L_n^{-1})^{(k)}(t_n) = \frac{1}{a_n^k}\, q_n^{(k)}(t_N) = x_{nk} - \frac{\alpha_n x_{Nk}}{a_n^k} \end{cases} \tag{14}$$

for $0 \leqslant k \leqslant p$.

The function $q_n \circ L_n^{-1}(t)$ is a polynomial of degree at most $2p+1$ whose derivatives up to order p at t_{n-1} and t_n are equal to the right-hand side of the expressions (14). Therefore $q_n \circ L_n^{-1}$ is a Hermite interpolating polynomial in $[t_{n-1}, t_n]$ whose existence and uniqueness is guaranteed [8]. From here it is deduced that $q_n(t)$ exists and is unique. The IFS given by (10) defines precisely one fractal interpolation function.

The functions $F_{nk}(t,x)$ defined by (12) verify the hypotheses of the Barnsley and Harrington theorem. By construction $\forall n = 2, 3, \ldots, N$ (13)

$$F_{nk}(t_0, x_{0k}) = x_{n-1,k} = F_{n-1,k}(t_N, x_{Nk}).$$

The theorem quoted assures the existence of $f \in \mathcal{C}^p$ such that $f^{(k)}$ is the FIF defined by the IFS $\{(L_n, F_{nk})\}_{n=1}^N$.

Consequently, $f^{(k)}$ is the fixed point of $T_k : \mathcal{F}_k \to \mathcal{F}_k$

$$(T_k g)(t) = F_{nk}(L_n^{-1}(t), g \circ L_n^{-1}(t)) \quad \forall t \in [t_{n-1}, t_n],$$

where $\mathcal{F}_k = \{g : [t_0, t_N] \to [c,d] \text{ cont.}; g(t_0) = x_{0k},\ g(t_N) = x_{Nk}\}$. $f^{(k)} \in \mathcal{F}_k$ and

$$f^{(k)}(t_0) = x_{0k}, \quad f^{(k)}(t_N) = x_{Nk}. \tag{15}$$

From (13) and (15)

$$\begin{aligned} f^{(k)}(t_n) &= F_{nk}(L_n^{-1}(t_n), f^{(k)}(L_n^{-1}(t_n))) = F_{nk}(t_N, f^{(k)}(t_N)) \\ &= F_{nk}(t_N, x_{Nk}) = x_{nk} \quad \forall n = 0, 1, \ldots, N \quad \forall k = 0, 1, \ldots, p. \qquad \square \end{aligned}$$

The above function f generalizes the Hermite functions as if $\alpha_n = 0\ \ \forall n = 1, 2, \ldots, N$ then $f \in \mathcal{C}^p$ and $f(t) = F_{n0}(L_n^{-1}(t), f \circ L_n^{-1}(t)) = q_n \circ L_n^{-1}(t)$ if $t \in [t_{n-1}, t_n]$. f is a polynomial of degree at most $2p+1$ in $I_n = [t_{n-1}, t_n]$ and consequently f is a Hermite function, $f \in H_\Delta^{p+1}[t_0, t_N]$.

Due to this result, a fractal interpolation function defined by the IFS (10) of the Theorem 3 will be called a Hermite fractal interpolation function (HFIF).

3. Bounds of the interpolation error

In the first place, the error committed by the substitution of a function $x(t)$ by the HFIF $f_{\boldsymbol{\alpha}}(t)$ with scale vector $\boldsymbol{\alpha}$ will be bounded.

Theorem 4 (*Ciarlet et al.* [4]). *Let $x(t) \in \mathcal{C}^r[0,1]$ with $r \geqslant 2p+2$, let Δ be any partition of $[0,1]$, $\Delta: t_0 < t_1 < \cdots < t_N$, and let $\varphi(t)$ be the unique interpolation of $x(t)$ in H_Δ^{p+1}, i.e., $x^{(l)}(t_n) = \varphi^{(l)}(t_n)$ for all $0 \leqslant n \leqslant N$, $0 \leqslant l \leqslant p$. Then, for all k with $0 \leqslant k \leqslant p+1$*

$$\|x^{(k)} - \varphi^{(k)}\|_\infty \leqslant C_k \, \|x^{(2p+2)}\|_\infty \, \|\Delta\|^{2p+2-k} \tag{16}$$

with

$$C_k = \frac{1}{2^{2p+2-2k} \, k! \, (2p+2-2k)!} \tag{17}$$

and $\|\Delta\| = \max_{0 \leqslant n \leqslant N-1} \{|t_{n+1} - t_n|\}$.

The Eq. (13) can be written as

$$\begin{cases} q_n^{(k)}(t_0) = a_n^k x_{n-1,k} - \alpha_n x_{0k}, \\ q_n^{(k)}(t_N) = a_n^k x_{nk} - \alpha_n x_{Nk} \end{cases} \tag{18}$$

for $0 \leqslant k \leqslant p$. The polynomials q_n can be considered as function of α_n and t, $q_n(\alpha_n, t)$.

Proposition 1. *The functions $q_n(\alpha_n, t)$ are indefinitely differentiable and the following inequalities are verified $\forall t \in [t_0, t_N]$ $\forall n = 1, 2, \ldots, N$.*

$$\left| \frac{\partial}{\partial \alpha_n} q_n(\alpha_n, t) \right| \leqslant D_0, \tag{19}$$

$$\left| \frac{\partial^{k+1}}{\partial \alpha_n \partial t^k} q_n(\alpha_n, t) \right| \leqslant D_k, \quad k = 1, 2, \ldots \tag{20}$$

with

$$D_0 = (2p+2) v d \tag{21}$$

$v = \max_{1 \leqslant n \leqslant N} \{\|P_n\|_\infty\}$ *and with P_n being the inverse of the coefficients matrix of the system* (18) *with unknown q_n, $d = \max_{o \leqslant k \leqslant p} \{|x_{0k}|, |x_{Nk}|\}$, $T = t_N - t_0$ and*

$$D_k = \frac{(2p+1)\, 2p \ldots \, (2p+1-k+1)}{T^k} \, v d \, (2p+2-k). \tag{22}$$

Proof. Define

$$q_n(t) = q_{0n} + q_{1n} \frac{t - t_0}{t_N - t_0} + q_{2n} \left(\frac{t - t_0}{t_N - t_0} \right)^2 + \cdots + q_{2p+1,n} \left(\frac{t - t_0}{t_N - t_0} \right)^{2p+1} \tag{23}$$

and let M_n be the coefficients matrix of the system (18) with unknown q_{jn} $(0 \leqslant j \leqslant 2p+1)$.

$$M_n (q_{jn})_{j=0}^{2p+1} = (c_{jn}(\alpha_n))_{j=0}^{2p+1}$$

with $c_{jn}(\alpha_n) = a_n^j x_{n-1,j} - \alpha_n x_{0j}$ for $0 \leqslant j \leqslant p$ and $c_{jn}(\alpha_n) = a_n^{j-p-1} x_{n,j-p-1} - \alpha_n x_{N,j-p-1}$ for $p+1 \leqslant j \leqslant 2p+1$. As the system admits a unique solution, the matrix M_n is nonsingular.

If $P_n = M_n^{-1}$, one has

$$(q_{jn})_{j=0}^{2p+1} = P_n (c_{jn}(\alpha_n))_{j=0}^{2p+1}. \tag{24}$$

The derivatives of c_{jn} with respect to α_n are:

$$\begin{cases} \frac{\partial c_{jn}}{\partial \alpha_n} = -x_{0j}, & 0 \leqslant j \leqslant p, \\ \frac{\partial c_{jn}}{\partial \alpha_n} = -x_{N,j-p-1}, & p+1 \leqslant j \leqslant 2p+1. \end{cases} \tag{25}$$

Define $d = \max_{0 \leqslant k \leqslant p} \{|x_{0k}|, |x_{Nk}|\}$ and $v_n = \|P_n\|_\infty = \max_{0 \leqslant i \leqslant 2p+1} \sum_{j=0}^{2p+1} |P_n^{ij}|$, $v = \max_{1 \leqslant n \leqslant N} \{v_n\}$.

From (24) and (25), it results

$$\left| \frac{\partial}{\partial \alpha_n} q_{jn}(\alpha_n, t) \right| \leqslant v_n \, d \leqslant v d$$

$\forall j = 0, 1, \ldots, 2p+1$ and so $\forall t \in [t_0, t_N]$ according to (23)

$$\left| \frac{\partial}{\partial \alpha_n} q_n(\alpha_n, t) \right| \leqslant \sum_{j=0}^{2p+1} \left| \frac{\partial}{\partial \alpha_n} q_{jn}(\alpha_n, t) \right| \leqslant (2p+2) \; vd.$$

If the expression (23) is differentiated k times, $1 \leqslant k \leqslant 2p+1$

$$q_n^{(k)}(t) = \sum_{r=k}^{2p+1} \frac{r(r-1)(r-2)\ldots(r-k+1)}{(t_N - t_0)^r} \; q_{rn}(t - t_0)^{r-k}$$

therefore, if $t \in [t_0, t_N]$

$$\left| \frac{\partial^{k+1}}{\partial \alpha_n \partial t^k} q_n(\alpha_n, t) \right| \leqslant \frac{(2p+1)2p \ldots (2p+1-k+1)}{T^k} \; vd \; (2p+1-k+1)$$

with $T = t_N - t_0$. $\square$

Consider the IFS (10) defined in the Theorem 3 and the mapping

$$T : J \times \mathcal{F} \to \mathcal{F},$$

$$(\boldsymbol{\alpha}, f) \to T_{\boldsymbol{\alpha}} f$$

with $J = [0, r] \times [0, r] \times \cdots \times [0, r] \subseteq \mathbb{R}^N$; $0 \leqslant r < 1$; r fixed and $I = [0, 1]$. For $t \in I_n = [t_{n-1}, t_n]$ define

$$T_{\boldsymbol{\alpha}} f(t) = F_{n0}^{\alpha_n}(L_n^{-1}(t), f \circ L_n^{-1}(t)) = \alpha_n f \circ L_n^{-1}(t) + q_n^{\alpha_n} \circ L_n^{-1}(t). \tag{26}$$

The superscript α_n represents the dependence regarding the vertical scale factor.

Proposition 2. *Let $f \in \mathcal{F}$, the following inequality holds*

$$\|T_{\boldsymbol{\alpha}} f - T_{\boldsymbol{\beta}} f\|_\infty \leqslant |\boldsymbol{\alpha} - \boldsymbol{\beta}|_\infty \, (\|f\|_\infty + D_0)$$

$|\boldsymbol{\alpha} - \boldsymbol{\beta}|_\infty = \max_{1 \leqslant n \leqslant N} \{|\alpha_n - \beta_n|\}$ *and D_0 defined in the Proposition* 1.

Proof. Let $f \in \mathcal{F}$, for each value $t \in I_n$

$$\begin{aligned}
&|T_{\boldsymbol{\alpha}} f(t) - T_{\boldsymbol{\beta}} f(t)| \\
&\quad = |\alpha_n f \circ L_n^{-1}(t) + q_n^{\alpha_n} \circ L_n^{-1}(t) - \beta_n f \circ L_n^{-1}(t) - q_n^{\beta_n} \circ L_n^{-1}(t)| \\
&\quad \leqslant |\alpha_n f \circ L_n^{-1}(t) - \beta_n f \circ L_n^{-1}(t)| + |q_n^{\alpha_n} \circ L_n^{-1}(t) - q_n^{\beta_n} \circ L_n^{-1}(t)|.
\end{aligned}$$

The first term verifies the inequality

$$|\alpha_n f \circ L_n^{-1}(t) - \beta_n f \circ L_n^{-1}(t)| \leqslant |\alpha_n - \beta_n| \, |f \circ L_n^{-1}(t)| \leqslant |\boldsymbol{\alpha} - \boldsymbol{\beta}|_\infty \, \|f\|_\infty. \quad (27)$$

To bound the second term, the mean-value theorem is applied. There exists $\xi_n \in (0, r)$ such that

$$q_n(\alpha_n, \tilde{t}) - q_n(\beta_n, \tilde{t}) = \frac{\partial q_n}{\partial \alpha_n}(\xi_n, \tilde{t})(\alpha_n - \beta_n)$$

and therefore,

$$|q_n^{\alpha_n} \circ L_n^{-1}(t) - q_n^{\beta_n} \circ L_n^{-1}(t)| \leqslant D_0 \, |\boldsymbol{\alpha} - \boldsymbol{\beta}|_\infty. \quad (28)$$

The result is obtained from inequalities (27)–(28). □

Proposition 3. *Let $f_{\boldsymbol{\alpha}}$, $f_{\boldsymbol{\beta}}$ be Hermite fractal interpolation functions with vertical scale vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The following inequality holds*:

$$\|f_{\boldsymbol{\alpha}} - f_{\boldsymbol{\beta}}\|_\infty \leqslant \frac{1}{1 - |\boldsymbol{\alpha}|_\infty} \, |\boldsymbol{\alpha} - \boldsymbol{\beta}|_\infty \, (\|f_{\boldsymbol{\beta}}\|_\infty + D_0).$$

Proof. By definition $f_{\boldsymbol{\alpha}}$, $f_{\boldsymbol{\beta}}$ are fixed points of $T_{\boldsymbol{\alpha}}$ and $T_{\boldsymbol{\beta}}$, respectively. Therefore $T_{\boldsymbol{\alpha}}(f_{\boldsymbol{\alpha}}) = f_{\boldsymbol{\alpha}}$, $T_{\boldsymbol{\beta}}(f_{\boldsymbol{\beta}}) = f_{\boldsymbol{\beta}}$. Applying the inequality (6) and the Proposition 2

$$\begin{aligned}
\|f_{\boldsymbol{\alpha}} - f_{\boldsymbol{\beta}}\|_\infty &= \|T_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}} - T_{\boldsymbol{\alpha}} f_{\boldsymbol{\beta}} + T_{\boldsymbol{\alpha}} f_{\boldsymbol{\beta}} - T_{\boldsymbol{\beta}} f_{\boldsymbol{\beta}}\|_\infty \\
&\leqslant \|T_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}} - T_{\boldsymbol{\alpha}} f_{\boldsymbol{\beta}}\|_\infty + \|T_{\boldsymbol{\alpha}} f_{\boldsymbol{\beta}} - T_{\boldsymbol{\beta}} f_{\boldsymbol{\beta}}\|_\infty \\
&\leqslant |\boldsymbol{\alpha}|_\infty \, \|f_{\boldsymbol{\alpha}} - f_{\boldsymbol{\beta}}\|_\infty + |\boldsymbol{\alpha} - \boldsymbol{\beta}|_\infty (\|f_{\boldsymbol{\beta}}\|_\infty + D_0).
\end{aligned}$$

From here

$$\|f_{\boldsymbol{\alpha}} - f_{\boldsymbol{\beta}}\|_\infty \leqslant \frac{1}{1 - |\boldsymbol{\alpha}|_\infty} \, |\boldsymbol{\alpha} - \boldsymbol{\beta}|_\infty \, (\|f_{\boldsymbol{\beta}}\|_\infty + D_0). \quad \square \quad (29)$$

Consequence: Setting $\boldsymbol{\beta} = \mathbf{0}$ in (29)

$$\| f_{\boldsymbol{\alpha}} - f_{\mathbf{0}} \|_\infty \leqslant \frac{1}{1 - |\boldsymbol{\alpha}|_\infty} |\boldsymbol{\alpha}|_\infty (\| f_{\mathbf{0}} \|_\infty + D_0). \tag{30}$$

As previously explained, $f_{\mathbf{0}}$ is a Hermite function that interpolates the data points. From here on, the case of equidistant nodes will be considered, $\|\Delta\| = h = t_n - t_{n-1}$ and $a_n = \frac{1}{N}$ (11).

Let $x(t)$ be the original function, $x(t) \in \mathcal{C}^r[0,1]$ $(r \geqslant 2p+2)$, with derivatives up to the order p prescribed at the nodes. One can bound $\| f_{\mathbf{0}} \|_\infty$ applying the Ciarlet et al. [4] theorem. Denoting $K_0 = C_0 L$; $L = \|x^{(2p+2)}\|_\infty$

$$\| f_{\mathbf{0}} \|_\infty \leqslant K_0 h^{2p+2} + \|x\|_\infty.$$

If $\|x\|_\infty = L_0$

$$\| f_{\boldsymbol{\alpha}} - f_{\mathbf{0}} \|_\infty \leqslant \frac{1}{1 - |\boldsymbol{\alpha}|_\infty} |\boldsymbol{\alpha}|_\infty (K_0 h^{2p+2} + L_0 + D_0). \tag{31}$$

Theorem 5. *Interpolation error bound*: *Let $x(t)$ be a function verifying $x(t) \in C^{2p+2}[0,1]$ and $L = \|x^{(2p+2)}\|_\infty$. Let $f_{\boldsymbol{\alpha}}$ be the $\mathcal{C}^p$ FIF defined in the theorem 3, $|\alpha_n| < a_n^p$. Then*

$$\|x - f_{\boldsymbol{\alpha}}\|_\infty \leqslant \frac{N^p}{N^p - 1} \left[K_0 h^{2p+2} + \frac{(L_0 + D_0)}{T^p} h^p \right],$$

where K_0 is the Ciarlet et al. constant $(K_0 = C_0 L)$, $L_0 = \|x\|_\infty$ and $T = t_N - t_0$.

Proof.

$$\|x - f_{\boldsymbol{\alpha}}\|_\infty \leqslant \|x - f_{\mathbf{0}}\|_\infty + \| f_{\mathbf{0}} - f_{\boldsymbol{\alpha}} \|_\infty$$

The first term can be bounded applying the theorem of Ciarlet et al. [4]

$$\|x - f_{\mathbf{0}}\|_\infty \leqslant K_0 h^{2p+2}. \tag{32}$$

In the second term the consequence of the Proposition 3 is used (31)

$$\| f_{\mathbf{0}} - f_{\boldsymbol{\alpha}} \|_\infty \leqslant \frac{1}{1 - |\boldsymbol{\alpha}|_\infty} |\boldsymbol{\alpha}|_\infty (K_0 h^{2p+2} + L_0 + D_0). \tag{33}$$

From (32)–(33)

$$\|x - f_{\boldsymbol{\alpha}}\|_\infty \leqslant \frac{1}{1 - |\boldsymbol{\alpha}|_\infty} \left[K_0 h^{2p+2} + |\boldsymbol{\alpha}|_\infty (L_0 + D_0) \right].$$

By hypothesis $|\boldsymbol{\alpha}|_\infty < \frac{1}{N^p} = \frac{h^p}{T^p}$ and, therefore, $\frac{1}{1 - |\boldsymbol{\alpha}|_\infty} \leqslant \frac{N^p}{N^p - 1}$, so the inequality above is transformed in

$$\|x - f_{\boldsymbol{\alpha}}\|_\infty \leqslant \frac{N^p}{N^p - 1} \left[K_0 h^{2p+2} + \frac{(L_0 + D_0)}{T^p} h^p \right]. \qquad \square \tag{34}$$

Following the theorem of Barnsley and Harrington, the derivatives $f^{(k)}$ of f are FIF corresponding to the IFS $\{(L_n(t), F_{nk}(t,x))\}_{n=1}^{N}$ $(k = 0, 1, \dots, p)$ with

$$F_{nk}(t,x) = N^k \alpha_n x + N^k q_n^{(k)}(t).$$

Consequently, the former results can be generalized up to pth derivative of f.

Proposition 4. *Let $f_{\boldsymbol{\alpha}}^{(k)}$, $f_{\boldsymbol{\beta}}^{(k)}$ be the kth derivatives $(0 \leqslant k \leqslant p)$ of $f_{\boldsymbol{\alpha}}$ and $f_{\boldsymbol{\beta}}$, respectively. Then*

$$\| f_{\boldsymbol{\alpha}}^{(k)} - f_{\boldsymbol{\beta}}^{(k)} \|_\infty \leqslant \frac{N^k |\boldsymbol{\alpha} - \boldsymbol{\beta}|_\infty}{1 - N^k |\boldsymbol{\alpha}|_\infty} (\| f_{\boldsymbol{\beta}}^{(k)} \|_\infty + D_k)$$

with D_k defined in Proposition 1 ((21),(22)).

Proof. Analogous to Proposition 3. □

Theorem 6. *Derivatives interpolation error bounds: Let $x(t)$ be a function verifying $x(t) \in C^{2p+2}[0,1]$ and $L = \|x^{(2p+2)}\|_\infty$. Let f_α the $\mathcal{C}^p$ FIF defined by the IFS* (10) *of Theorem* 3 *with $h = t_n - t_{n-1}$ $\forall n = 1, 2, \dots, N$. Let $s = s(N)$ such that $0 < s < 1$ and $|\boldsymbol{\alpha}|_\infty \leqslant \frac{1}{N^{p+s}}$ then*

$$\|x^{(k)} - f_{\boldsymbol{\alpha}}^{(k)}\|_\infty \leqslant \frac{N^{p+s-k}}{N^{p+s-k} - 1} \left[K_k h^{2p+2-k} + \frac{(L_k + D_k)}{T^{p+s-k}} h^{p+s-k} \right]$$

for $0 \leqslant k \leqslant p$, being K_k the constant of the Ciarlet et al. theorem $(K_k = C_k L)$, $L_k = \|x^{(k)}\|_\infty$, $T = t_N - t_0$, D_k defined by (21) *and* (22).

Proof. By hypothesis $|\boldsymbol{\alpha}|_\infty < \frac{1}{N^p}$. Since $\frac{1}{N^{p+x}} \to \frac{1}{N^p}$ as $x \to 0^+$, there exists $s = s(N)$ such that $0 < s < 1$ and $|\boldsymbol{\alpha}|_\infty \leqslant \frac{1}{N^{p+s}}$. The rest is analogous to the Theorem 5. □

Consequence: Clearly, Theorem 6 implies that for sequences $\Delta_m = \{0 = t_0^{(m)} < t_1^{(m)} < \cdots < t_{N_m}^{(m)} = 1\}$, $m = 0, 1, 2, \dots$ of partitions with $h_m \to 0$, if the partial derivatives of the polynomials are uniformly bounded, the corresponding fractal interpolation functions converge to $x(t)$ in the C^{p-1} norm on $I = [0, 1]$.

4. Conclusions

The present paper proposes a method of fractal differentiable interpolation for the approximation of functions and the numerical processing of experimental signals. The theorem of Barnsley and Harrington provides the construction of a generalization of the Hermite functions space. With the help of some results concerning osculatory polynomials, interpolation error estimates have been obtained, assuming some hypotheses on the original function.

As a consequence, the uniform convergence of Hermite fractal functions to the original function and its first derivatives when the partition diameter tends to zero is deduced.

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